

# Collapse instability of solitons in the nonpolynomial Schrödinger equation with dipole-dipole interactions

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**Abstract.** A model of the Bose-Einstein condensate (BEC) of dipolar atoms, confined in a combination of a cigar-shaped trap and optical lattice acting in the axial direction, is studied in the framework of the one-dimensional (1D) nonpolynomial Schrödinger equation (NPSE) with additional terms describing long-range dipole-dipole (DD) interactions. The NPSE makes it possible to describe the collapse of localized modes, which was experimentally observed in the self-attractive BEC confined in tight traps, in the framework of the 1D description. We study the influence of the DD interactions on the dynamics of bright solitons, especially as concerns their collapse-induced instability. Both attractive and repulsive contact and DD interactions are considered. The results are summarized in the form of stability/collapse diagrams in a respective parametric space. In particular, it is shown that the attractive DD interactions may prevent the collapse instability in the condensate with attractive contact interactions.

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## 1. Introduction

In the mean-field approximation, the dynamics of Bose-Einstein condensates (BECs) obeys the 3D Gross-Pitaevskii equation (GPE) [1], from which an effective 1D equation can be derived, in various settings, for condensates trapped in prolate traps [2]-[7]. In the simplest case, which corresponds to a sufficiently low BEC density, the reduction of the 3D equation for the BEC trapped in the “cigar-shaped” configuration leads to the one-dimensional (1D) cubic nonlinear Schrödinger equation (NLSE) [8]. The main restriction on the use of this equation is its inability to describe the onset of collapse of localized states, which was theoretically predicted in the 3D setting and experimentally observed in the self-attractive BEC [9], [10]. However, without imposing the constraint of a very low density, the reduction of the 3D GPE leads to the 1D equation with a *nonpolynomial* nonlinearity, alias the nonpolynomial Schrödinger equation (NPSE) [4, 11]. The 1D NPSE with the attractive sign of the nonlinearity enables the description of the collapse dynamics and produces results which are corroborated by direct simulations of the underlying 3D GPE [12].

On the other hand, it is known that the mean-field description of BECs trapped in a very deep optical-lattice (OL) potential can be well described by the corresponding discrete equations. In particular, discrete forms of the 1D GPE with the cubic nonlinearity [13, 14, 15], and of the 1D NPSE [16] have been studied in detail. Basic features of the 1D continual equations describing BECs trapped in deep OLs find their counterparts in the discrete models. In particular, the ability of the continual 1D NPSE to capture the onset of the collapse is also shared by the corresponding discrete equations.

A new variety of the BEC dynamics, which is dominated by long-range (nonlocal) interactions, occurs in dipolar condensates, which may be formed by magnetically polarized <sup>52</sup>Cr atoms [17], dipolar molecules [18], or atoms with electric dipole moments induced by an external field [19]. A review of dynamical effects produced by the dipole-dipole (DD) interactions in condensates can be found in Ref. [20]. In particular, conditions for the stability of the trapped dipolar BEC against collapse were studied in detail [21]. A possibility of the creation of 2D solitons in dipolar condensates was predicted too. Namely, isotropic solitons [22] and solitary vortices [23] may exist if the sign of the dipole-dipole (DD) interaction is inverted by means of rapid rotation of the dipoles [24]. On the other hand, stable anisotropic solitons can be supported by the ordinary DD interaction, if the dipoles are polarized in the 2D plane [25]. Solitons supported by nonlocal interactions were also predicted and realized in optics, making

use of the thermal nonlinearity [26].

A natural extension of the consideration of the dipolar BEC includes the OL potential, which, in the discrete limit, leads to the model with the long-range DD interactions between lattice sites [27, 28, 29]. Very recently, 1D solitons have been studied in the framework of the continual cubic GPE, assuming the competition of local and nonlocal DD interactions, with or without the OL potential [30]. In the discrete limit, 1D solitons supported by the DD interactions were recently studied too, in models with both the cubic [27] and nonpolynomial [28] onsite nonlinearity. In particular, the latter work predicts a possibility of suppressing the collapse by means of the long-range DD forces. The main purpose of the present work is to investigate the influence of the DD interactions on the collapse dynamics in the 1D continual NPSE, which is the most adequate setting for the study of the onset and suppression of the collapse in the dipolar BEC loaded into a cigar-shaped trap.

The paper is structured as follows. The model equation which includes the nonpolynomial local nonlinearity and nonlocal DD interactions, is formulated in Section II, where we also outline numerical techniques that we use in this work. Focusing on the study of fundamental bright solitons, we report basic results for their existence and stability in Section III. The core part of paper is Section IV, where we study the influence of the DD interaction on the solitons' collapse. The paper is concluded by Section V.

## 2. The model

The dynamics of a BEC at zero temperature is accurately described by the 3D Gross-Pitaevskii equation (3D GPE) [1]. When the condensate is confined by a harmonic potential with frequency  $\omega_\perp$  and respective length  $a_\perp = (\hbar/m\omega_\perp)^{1/2}$  in the transverse plane, and by generic potential  $V(z)$  in the axial direction, it was shown in Ref. [4] that the corresponding 3D GPE can be reduced to the 1D NPSE for wave function  $\psi(z, t)$ , which is subject to normalization  $\int_{-\infty}^{+\infty} |\psi(z)|^2 dz = 1$ . The equation includes the OL potential, with depth  $V_0$  and wavenumber  $K$ , and the long-range DD (cf. the 1D equations introduced in Refs. [27, 30]):

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial z^2} + V_0 \sin^2(Kz)\psi + \frac{1 - \frac{3}{2}\gamma|\psi|^2}{\sqrt{1 - \gamma|\psi|^2}}\psi + G\psi(z) \int_{-\infty}^{+\infty} \frac{|\psi(z')|^2}{|z - z'|^3} dz'. \quad (1)$$

Here,  $\gamma = -2Na_s\sqrt{m\omega_\perp/\hbar}$  is the effective strength of the local interaction, with  $N$  the total number of atoms in the condensate, and  $a_s$  the scattering length of atomic collisions ( $a_s < 0$  corresponds to attraction) [4]. Further,  $G = g(1 - 3\cos^2\theta)$  is the coefficient which defines the DD interaction, where  $g$  is a positive coefficient, and  $\theta$  the angle between the  $z$  axis and the orientation of the dipoles.

Replacing wave function  $\psi$  by  $f \equiv \sqrt{|\gamma|}\psi$ , we transform Eq.(1) into a normalized

form,

$$i \frac{\partial f}{\partial t} = -\frac{1}{2} \frac{\partial^2 f}{\partial z^2} + V_0 \sin^2(Kz) f + \frac{1 - \frac{3}{2} \aleph |f|^2}{\sqrt{1 - \aleph |f|^2}} f + \Gamma f(z) \int_{-\infty}^{+\infty} \frac{|f(z')|^2}{|z - z'|^3} dz', \quad (2)$$

where  $\aleph = \text{sgn}(\gamma)$  is the sign of the local interaction ( $\aleph = +1$  corresponds to the attraction), and  $\Gamma \equiv G/|\gamma|$  measures the relative strength of the DD and contact interactions.

One case of obvious interest for the 1D setting is that when the dipoles are aligned with the  $z$  axis, i.e.,  $\theta = 0$  and  $\Gamma = -2g/|\gamma| < 0$ , which means that the DD interaction is attractive. Another relevant case corresponds to the dipoles oriented perpendicular to the  $z$  axis, i.e.  $\theta = \pi/2$  and  $\Gamma = g/|\gamma| > 0$ , which implies the repulsive DD interaction. Thus, the sign of  $\Gamma$  in the present model defines the character of the DD interaction.

Stationary solutions to Eq. (2), with chemical potential  $\mu$ , are sought for as  $f = u \exp(-i\mu t)$ , with function  $u$  satisfying the stationary equation,

$$\mu u = -\frac{1}{2} \frac{\partial^2 u}{\partial z^2} + V_0 \sin^2(Kz) u + \frac{1 - \frac{3}{2} \aleph |u|^2}{\sqrt{1 - \aleph |u|^2}} u + \Gamma u(z) \int_{-\infty}^{+\infty} \frac{|u(z')|^2}{|z - z'|^3} dz'. \quad (3)$$

To present the results for soliton families, we will use the norm defined by

$$P = \int_{-\infty}^{+\infty} |u(z)|^2 dz \quad (4)$$

(according to the above definitions,  $P$  is identical to  $|\gamma|$ , but parameter  $\gamma$  is not used below).

Experimentally adjustable coefficients in this model are the relative strength of the DD/contact interactions,  $\Gamma$ , and the norm of the stationary wave function,  $P \sim N |a_s| \sqrt{m\omega_\perp/\hbar}$  [14, 16, 27]. In particular,  $P \sim 1$  corresponds to  $\sim 1000$  atoms in the  $^{52}\text{Cr}$  condensate [14, 16]. Actually,  $\Gamma$  can be made both positive and negative, and its absolute value may be altered within broad limits by means of the Feshbach resonance (without the application of the Feshbach resonance,  $|\Gamma| \simeq 0.15$  in the condensate of chromium atoms) [17, 31].

Stationary equation (3) was solved numerically by means of an algorithm based on the shooting method. We restrict the analysis to fundamental solitons with a single maximum. Therefore, we chose the following boundary conditions in the shooting procedure: the first derivative of the wave function must be zero at the point of the maximum amplitude, and the wave function must decay exponentially at infinity. Note that the integral term in Eq. (3) has the form of a convolution [32],

$$\int_{-\infty}^{+\infty} V(z - z') |u(z')|^2 dz' \equiv V(z) \times |u(z)|^2, \quad (5)$$

hence its Fourier transform can be calculated as a product of the Fourier images of the interaction potential and local density,  $|u(z)|^2$ . To calculate the integral term more accurately, we adopted an iterative shooting procedure. For the first iteration, we assumed that the potential of the interaction in the Fourier space is approximated by a constant [32]. In each next iteration, we used the previous solution to calculate the convolution integral numerically (5).

Time-dependent equation (2) was solved by means of the split-step Fourier method [33]. To that end, Eq. (2) was cast into an operator form,

$$\frac{\partial f}{\partial t} = i(\hat{V} + \hat{D} + \hat{L} + \hat{N})f, \quad (6)$$

where  $\hat{V} \equiv -V_0 \sin^2(Kz)$ ,  $\hat{D} \equiv -\Gamma \int_{-\infty}^{+\infty} |f(z')|^2 / |z - z'|^3 dz'$ ,  $\hat{L} \equiv \partial^2 / (2\partial z^2)$ , and  $\hat{N} \equiv -(1 - \aleph |f|^2)^{-1/2} [1 - (3/2)\aleph |f|^2]$ . The split-step Fourier method was implemented through independent actions of all of the operators,  $\hat{V}$ ,  $\hat{D}$ ,  $\hat{L}$  and  $\hat{N}$ , within time step  $\tau$  of the integration scheme,  $f(t + \tau, z) = e^{-i\tau\hat{D}} e^{-i\tau\hat{V}} e^{-i\tau\hat{L}} e^{-i\tau\hat{N}} f(t, z)$  (we fixed  $\tau = 10^{-4}$ ).

### 3. Fundamental solitons

Stationary wave function  $u(z)$  was found from the solution of stationary equation (3). In the case of the noninteracting condensate ( $\gamma = 0$ ), and without the DD interactions ( $\Gamma = 0$ ), equation (3) is tantamount to the Mathieu equation,

$$\frac{d^2 u}{dy^2} - [2q \cos(2y) - p] u = 0, \quad (7)$$

where  $y \equiv Kz$ ,  $q \equiv -V_0/2K^2$ , and  $p \equiv 2(q - 1/K^2 + \mu/K^2)$ , hence the well-known bandgap diagram for the Mathieu equation (see, e.g., Ref. [34]) can be used. That diagram, translated into the notation adopted above, is displayed in Fig. 1. The nonlinear localization of matter waves in the form of gap solitons may occur in gaps of the linear spectrum. In the case of the local attractive nonlinearity, fundamental solitons populate the semi-infinite gap.

Figure 2 displays typical examples of fundamental solitons, with equal values of the norm, which were obtained as numerical solutions to Eq. (3) in cases of zero, attractive, and repulsive DD interactions. In the presence of the attractive DD interaction, the solitons are narrower and feature a higher amplitude, while in the case of the DD repulsion, they are wider, and have a lower amplitude than the respective soliton in the absence of the DD interaction. The peculiarity of the NPSE in the case of the attractive contact interaction is that the amplitude of the soliton is limited by a critical value. The collapse setting in when the amplitude attains that value [4, 12]. In contrast to that, in the ordinary 1D GPE the amplitude may grow indefinitely without causing the collapse.

If the local interaction is repulsive ( $\aleph = -1$ ), fundamental solitons in the NPSE may only be supported by the attractive DD interaction, which must dominate over the contact repulsion. In that case, the difference between the NPSE and ordinary

1D GPE is not significant. In particular, as the collapse instability is absent, unstable fundamental solitons can only evolve into breathers. The situation in the case of the repulsion is similar to that in the case of the discrete NPSE [27].

#### 4. The influence of the dipole-dipole interactions on the collapse

In this section we focus on the dynamics of fundamental solitons in the presence of the attractive contact interaction ( $\aleph = +1$ ) and nonlocal DD of either sign. The collapse occurs in this case, as well as in the respective discrete setting [27].

A global characteristic of soliton families is the  $P(\mu)$  dependence, the norm versus the chemical potential. In Fig. 3 we display the  $P(\mu)$  curves obtained for different values of  $\Gamma$ . The stability of the soliton may be assessed according to two different conditions:  $dP/d\mu \leq 0$  [the *Vakhitov-Kolokolov* (VK) criterion], and the absence of eigenvalues for small perturbations with positive real parts (the spectral condition), see Ref. [35] and references therein.

In the presence of the attractive contact interaction, the  $P(\mu)$  dependences feature two different regions, one where the VK criterion is fulfilled, and another one, where it is violated. The region where the VK criterion holds expands/shrinks with the increase of the strength of the attractive/repulsive DD interactions. These results are in qualitative agreement with findings for on-site solitons in the discrete version of 1D NPSE [27].

The spectral stability condition was examined by numerically computing the corresponding eigenvalues, using linearized equations for small perturbations. It has been found that the spectral condition is *violated* in the *entire parameter space* which was explored, see Fig. 4. The eigenvalue spectra are quite similar to those found for inter-site solitons in the discrete NPSE with the DD interaction, at large values of the inter-site coupling constant  $C$ , which corresponds to the quasi-continuum limit [27]. Note that the spectra for solitons in the NPSE exhibit an abrupt (quasi-exponential) growth of real parts of the eigenvalues [see Fig. 4(a)], which does not happen in the respective GPE with the cubic nonlinearity, cf. Fig. 4(b). An interesting fact revealed by the numerical analysis is that the threshold of the abrupt growth of the real part of the complex eigenvalues corresponds to the onset of the collapse instability of the solitons. Indeed, direct simulations of Eq. (2) show that unstable solitons with  $\mu$  higher than the threshold value evolve into robust localized breathers [Fig. 5(a)], while the solitons with  $\mu$  taken below the threshold exhibit the collapse instability, which manifests itself through simultaneous decrease of the soliton's width and growth of the amplitude towards the limit value, as seen in Fig. 5(b). Points of the onset of the collapse instability are also marked in Fig. 3. It is worthy to note that the instability sets in earlier than the norm attains the maximum values (full curves  $P(\mu)$  could be drawn in spite of the instability, as they were obtained from the numerical solution of stationary equation (3)).

With the increase of the strength of the attractive DD interaction, the threshold value of  $\mu$  becomes lower, hence the region in the parameter space where unstable

solitons do not collapse but rather evolve into breathers expands. These results are presented as the *collapse diagram* in the  $(\Gamma, \mu)$  plane (see Fig. 6), where the border line separates the collapse region from that where the formation of breathers takes place. Qualitatively, the nearly linear dependence of the threshold on  $\Gamma(\mu)$  can be understood, assuming that the nonlocality range is very large, covering the entire soliton. Indeed, in that case the last term in Eq. (3) is proportional to  $\Gamma Pu(z)$ , which implies a linear shift of  $\mu$ .

Thus, the collapse instability of the fundamental solitons may be suppressed by using sufficiently strong attractive DD interactions. This prediction of the stability analysis based on the computation of the eigenvalues was fully corroborated by direct simulations of Eq. (2).

## 5. Conclusion

The purpose of this work was to achieve a better understanding on the influence of the nonlocal DD (dipole-dipole) interactions on the stability and collapse of localized nonlinear modes in the BEC trapped in a combination of a tight transverse parabolic potential and a relatively loose periodic OL potential acting in the axial direction. To this end, we have introduced the model based on the one-dimensional NPSE (nonpolynomial Schrödinger equation), which includes the contact and DD interaction terms, as well as the OL potential. Both attractive and repulsive signs of the contact and DD interactions were considered. The analysis was focused on fundamental solitons in the semi-infinite gap. While all the stationary solitons are unstable, an essential conclusion is that the *attractive* DD interactions may prevent the collapse instability of the fundamental solitons, replacing the onset of the collapse by the transformation of the solitons into robust breathers. This general result is consistent with findings recently reported for the 1D discrete version of the NPSE.

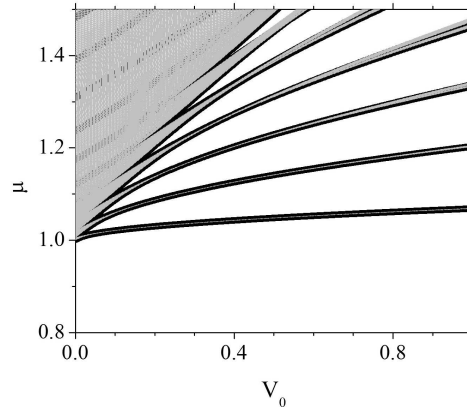
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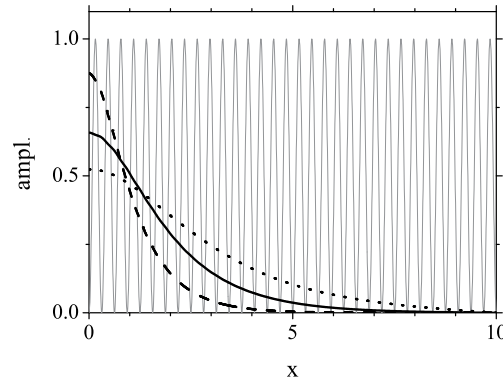
## Figures and figure captions

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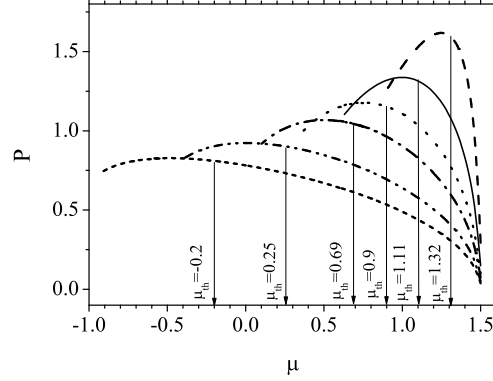
**Figure 1.** The bandgap diagram for the linearized model. Shaded regions depict the Bloch bands, separated by the gaps, where gap solitons can exist in the nonlinear system. Solid lines depict band edges, which correspond to periodic Bloch waves. The semi-infinite gap is located below the lowest band.



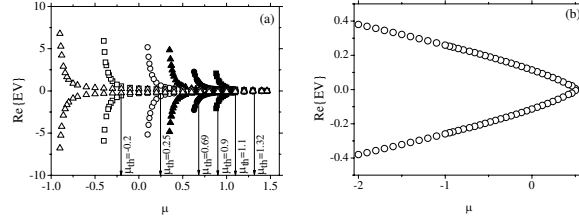
**Figure 2.** Fundamental solitons for three different cases: in the absence of the DD interaction ( $\Gamma = 0$ ,  $\mu = 1.254$ , the solid line); in the presence of the attractive DD interaction ( $\Gamma = -0.5$ ,  $\mu = 0.8$ , the dashed line); in the presence of the repulsive DD interaction ( $\Gamma = 0.35$ ,  $\mu = 1.4$ , the dotted line). In all the cases, the strength of the optical-lattice potential (shown by gray lines) is  $V_0 = 1$ , and the soliton's norm is  $P = 1.173$ .

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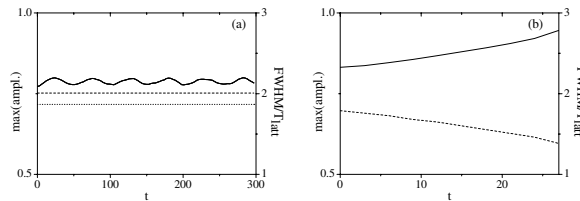




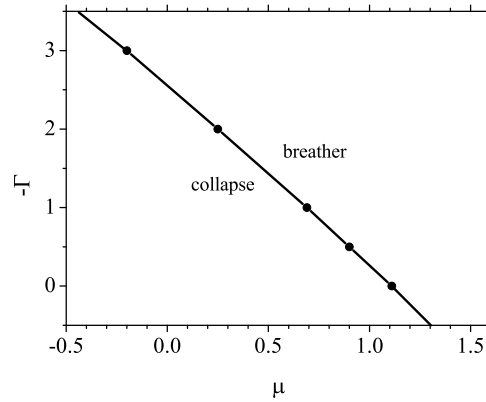
**Figure 3.**  $P(\mu)$  diagrams for fundamental solitons in the case of attractive contact and repulsive, zero, or attractive DD interactions:  $\Gamma = 0.5$  (the dashed line),  $\Gamma = 0$  (the solid line),  $\Gamma = -0.5$  (the dotted line),  $\Gamma = -1$  (the dash-dotted line),  $\Gamma = -2$  (the dashed-dotted-dotted line),  $\Gamma = -3$  (the short-dash line). In all the cases,  $V_0 = 1$ . Values of  $\mu_{th}$  shown in the plots designate thresholds of the collapse instability.



**Figure 4.** The real parts of eigenvalues for small perturbations around the fundamental soliton versus  $\mu$ , for the NPSE (a) and cubic GPE (b). The contact and DD interactions are both attractive in this case. Parameters are  $V_0 = 1$ , and (a)  $\Gamma = 0.5$  (squares),  $\Gamma = 0$  (circles),  $\Gamma = -0.5$  (triangles),  $\Gamma = -1$  (empty circles),  $\Gamma = -2$  (empty squares),  $\Gamma = -3$  (empty triangles), or (b)  $\Gamma = -0.5$ . Values of  $\mu_{th}$  in (a) designate the threshold of the collapse instability.



**Figure 5.** Illustration of the evolution of unstable fundamental solitons into breathers, or onset of the collapse. In the former case (a) the oscillating amplitude of the breather, and in the latter case (b) the monotonously growing soliton amplitude are plotted by the solid curves. Dashed and dotted lines in (a) depict the smallest and largest values of the FWHM (full width at half maximum) for the breather, while in (b) the FWHM of the collapsing soliton is depicted by the dashed line. In all the cases the unit of FWHM is the period of the optical lattice period,  $T_{latt}$  and  $V_0 = 1$ .



**Figure 6.** The collapse diagram for fundamental solitons in the NPSE with the attractive sign of both the contact and DD interactions ( $V_0 = 1$ ). The plotted line separates parameter regions where solitons evolve into breathers, and regions where solitons suffer the collapse.

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